1. THE REAL PROJECTIVE PLANE

§1.1. The History of Perspective

Projective Geometry grew out of the struggle by Renaissance artists to make their paintings more realistic by introducing depth.

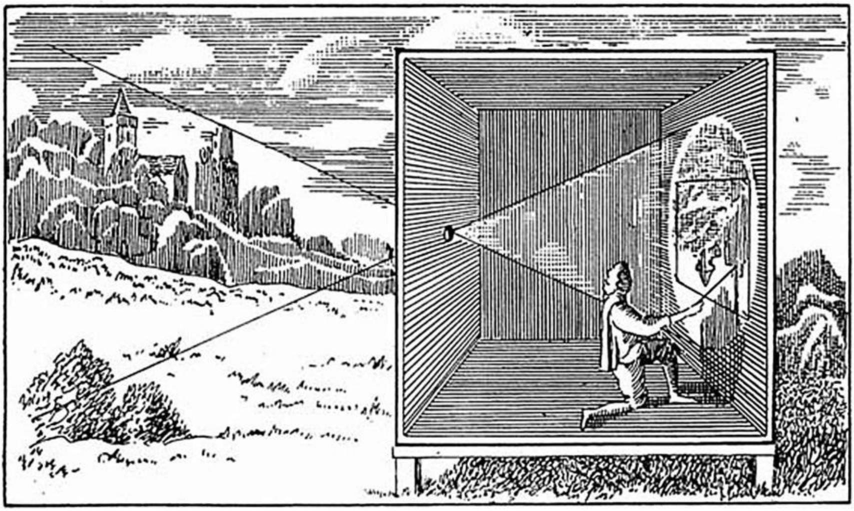
Art in ancient times was very 2-dimensional, such as the Egyptian art that generally depicted the human form side on.

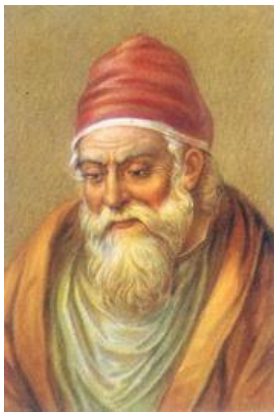
There are several techniques in art to give depth to a painting, such as making distant objects smaller and making their outlines less distinct. But the technique of perspective is by far the most important.

The Renaissance artist, Filippo Brunelleschi (1377-1446) is considered to be the first artist to really master perspective, and to introduce the concept of a vanishing point.

The basic problem in converting a 3-dimensional scene to a 2-dimensional canvas is that the human eye perceives the scene as 3-dimensional and that gets in the way of capturing the 3-dimensional nature of what is being painted.

Certain optical instruments, such as the *camera obscura*, were developed which projected the 3-dimensional scene onto a 2-dimensional screen. The artist simply had to make a faithful copy of what was projected onto the screen.





The Greek geometer, Euclid (4th century BC) Geometry began the systematic study of geometry, both two and three dimensional, and introduced the concept of the Euclidean Plane. But more fundamental than the Euclidean Plane is the Affine Plane and it is the Affine Plane that provides the foundation for Projective Geometry.

§1.2. The Real Affine Plane

The Euclidean Plane involves a lot of things that can be measured, such as distances, angles and areas. This is referred to as the *metric structure* of the Euclidean Plane. But underlying this is the much simpler structure where all we have are points and lines and the relation of a point lying on a line (or equivalently a line passing through a point). This relation is referred to as the **incidence structure** of the Euclidean Plane.

The **Real Affine Plane** is simply the Euclidean Plane stripped of all but the incidence structure. This eliminates any discussion of circles (these are defined in terms of distances) and trigonometry (these need measurement of angles). It might seem that there's nothing left but this isn't the case.

There is, of course, the concept of **collinearity**. Three or more points are **collinear** if there is some line that passes through them all. Certain theorems of affine geometry state that if there's such and such a configuration and certain triples of points are collinear then a certain other triple is collinear. Another affine concept is that of **concurrence** – three or more lines passing through a single point.

Then there's the concept of **parallelism**, though we can't define it in terms of lines having a constant distance between them or lines having the same slope.

We say that two lines h, k are **parallel** in the Real Affine Plane if either they coincide or they don't intersect (meaning that no point lies on them both).

Here are two basic properties of the Real Affine Plane:
(1) Given any two distinct points there is *exactly one* line passing through both.
(2) Given any two distinct lines there is *at most one* point lying on both.

There's an obvious similarity between these two properties. But while we can say "exactly one" in property (1) the best we can do in property (2) is "at most one" because there is no common point when the lines are parallel.

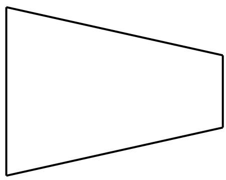
This is reminiscent of the situation with the real number system, where every non-zero real number has at most two square roots. Why not *exactly two*? The reason is that negative numbers don't have any square roots.

This was considered to be a defect of the real number system and so imaginary numbers were invented to provide square roots for negative numbers. The field of real numbers was expanded to form the complex numbers, and for this system we can say that *every* non-zero number has *exactly two* square roots.

We do the same thing with the Real Affine Plane. We invent 'imaginary' points where parallel lines can meet. We call these extra points, **ideal points**. And we invent an 'imaginary' line called the **ideal line**, and specify that all the ideal points lie on this line. The Real Affine Plane is thereby extended to the Real Projective Plane. In the Real Projective Plane there are no longer such things as distinct parallel lines since every pair of distinct lines now intersect in exactly one point. This then mirrors exactly the fact that every pair of distinct points lie on exactly one line.

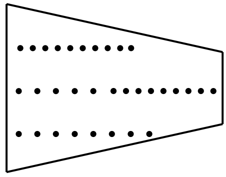
§1.3. Intuitive Construction

We start with the Real Affine Plane.

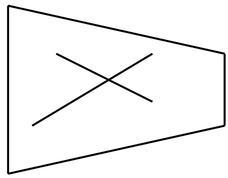


This trapezium is merely a representative picture of the affine plane. If we drew an accurate picture it would cover the whole page, and then some! There'd be no space for any extra points, let alone the text. You should view this as a perspective view of a very large rectangle and then pretend that it extends infinitely far in all directions.

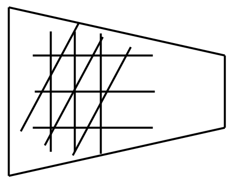
We refer to the points on the Real Affine Plane as **ordinary points**.



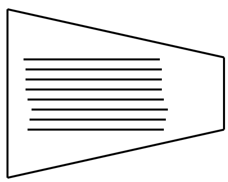
We call the lines on the Real Affine Plane **ordinary lines**.



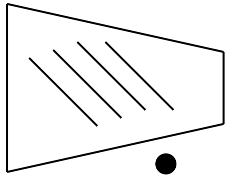
We sort these ordinary lines into parallel classes.



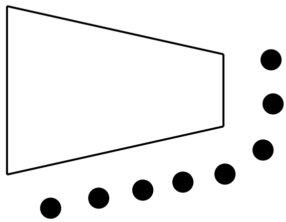
A **parallel class** consists of a line together with all lines parallel to it.



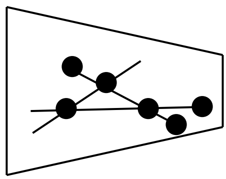
For each parallel class we invent a new point, called an **ideal point**.



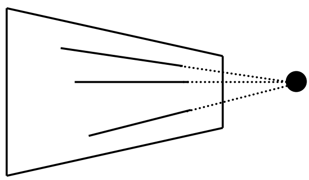
These ideal points don't lie on the Real Affine Plane. Where are they then? The answer is simply "in our minds". However, to assist our imagination, we can put these ideal points on our diagram outside of the shape that represents the Real Affine Plane.



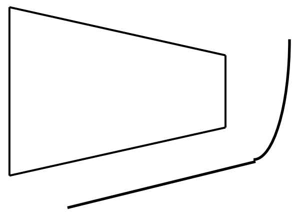
As well as ordinary points lying on ordinary lines in the usual way



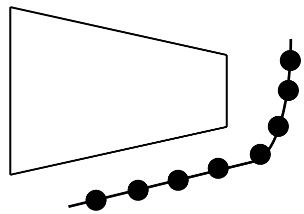
we decree that all lines in a given parallel class (and no others) pass through the corresponding ideal point.



We also invent a new line called the **ideal line**



and decree that this line passes through all the ideal points (and no others).

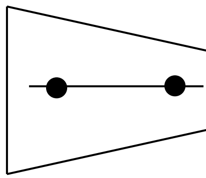


The resulting geometry is called the **Real Projective Plane**. It contains all of the Real Affine Plane, as well as the ideal points and the ideal line. Any theorem that we can prove for the real projective plane will be true for the Real Affine Plane simply by taking the special case of ordinary points and lines.

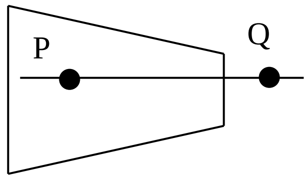
§1.4. The Real Projective Plane is Complete

Theorem 1A: In the real projective plane:
(1) any two distinct points lie on exactly one line.
(2) any two distinct lines intersect in exactly one point.

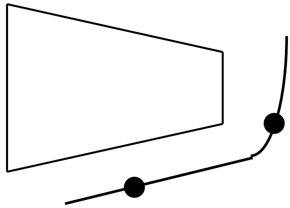
Proof: (1) **Case I:** two ordinary points lie on an ordinary line, as in the affine plane.



Case II: an ordinary point P and an ideal point Q lie on the line through P parallel to the lines in the parallel class corresponding to Q. (Remember that in Euclidean Geometry there's a unique line which passes through a given point and is parallel to a given line.)

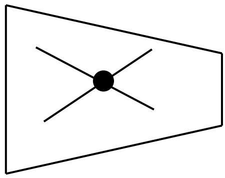


Case III: Two ideal points lie on the ideal line.

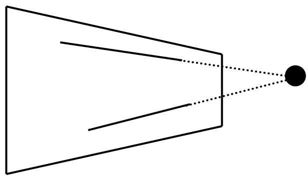


In all these cases there is just the one line passing through the two points, ordinary or ideal.

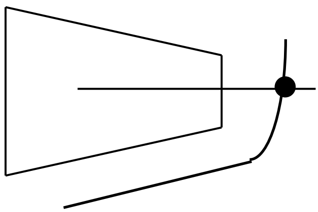
(2) **Case I: Two ordinary lines:**
If these lines are non-parallel they intersect in an ordinary point.



If they’re parallel they intersect in the ideal point that corresponds to their parallel class.



Case II: An ordinary line and the ideal line:
An ordinary line intersects the ideal line in the ideal point corresponding to the parallel class in which it lies.



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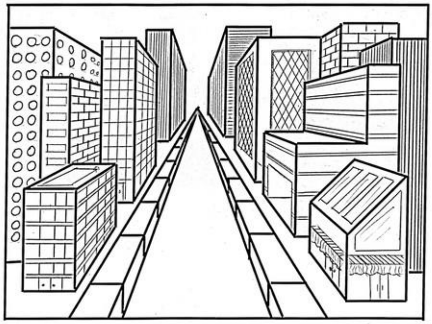
In all these cases there is just the one point lying on both lines, ordinary or ideal.

§1.5. The Artist's View

Renaissance artists had no problem with the concept of parallel lines meeting at a point. This happens all the time in a perspective drawing.



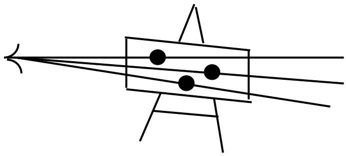
Consider what a Renaissance artist did when she sketched a scene. "She?" you may ask, "were there any female Renaissance painters?"

Indeed there were, though most of them were portrait painters where questions of perspective were not so relevant.

But Nelli Plautilla (1524-1588) was a female Renaissance painter, the first such in Florence. She was a Dominican nun and so it may not have been so easy for her to paint from nature.

But suppose that Nelli had been allowed to assemble a random collection of saints for the following painting, with those buildings in the background. There's an interesting mixture of foreground and background in this painting.

Every ray emanating from her eye (it was once thought that sight traveled from the eye to the objects being seen) corresponds to a single point on her canvas.

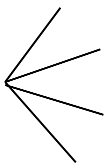


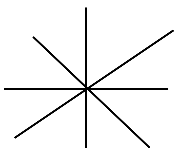
This leads to the next way of thinking about the Real Projective Plane. Consider the 3-dimensional Euclidean space, $\mathbb{R}^3$.

A **projective point** is a line through the origin.
A **projective line** is a plane through the origin.

It might seem strange at first to call a line a point and to call a plane a line but to the artist, a line in 3-dimensional space, passing through her viewpoint, is what she depicts as a single point on her canvas. This is not completely true, in that an artist (unless she has eyes in the back of her head) only depicts rays (half-lines) not whole lines.

rays





lines

The Real Projective Plane is defined as the set of all projective points and all projective lines. It can be thought of as a sort of porcupine in 3-dimensional space, bristling with lines going in all directions through the origin. To complete the description we must define incidence, that is, we must explain what it means for a projective point to lie on a projective line.

- A projective point P (line through the origin) **lies on** a projective line *h* (plane through the origin) if, when considered as a line, it lies on *h*, considered as a plane.
- A projective line *h* **passes through** a projective point P if P, considered as a line, lies on *h*.
- Three (or more) projective points are **collinear** if they lie on a common projective line.
- Three (or more) projective lines are **concurrent** if they pass through the same projective point.

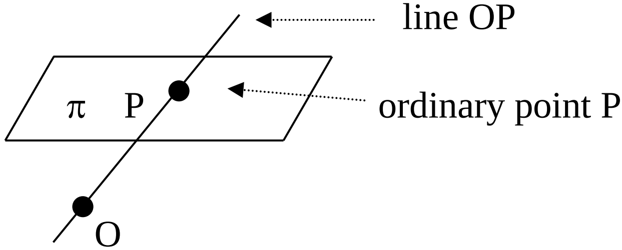
Theorem 1B: In the Real Projective Plane:
(1) any two distinct projective points lie on exactly one projective line.
(2) any two distinct projective lines intersect in exactly one projective point.
Proof: Translated into ordinary terms these state that:
(1) any two lines through the origin lie on exactly one plane through the origin and
(2) any two planes through the origin intersect in exactly one line through the origin.

Both of these are clearly true statements for 3-dimensional Euclidean Space. ☺✋

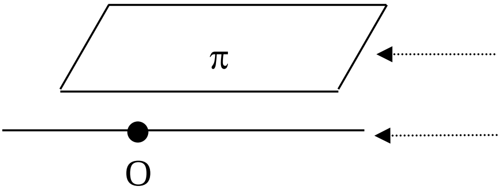
§1.6. Embedding the Real Affine Plane

It's all very well to have constructed a geometry whose incidence structure is complete, but is it really the same structure that we created earlier, by extending the Real Affine Plane by adding ideal points? After all, with this new version, which projective points are the ideal points? With this bunch of lines through the origin they all seem pretty much the same.

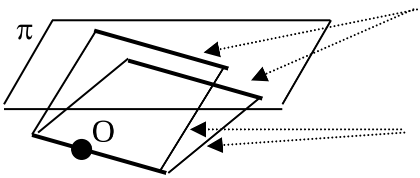
Well, suppose we take an ordinary Real Affine Plane π in $\mathbb{R}^3$, one that doesn't pass through the origin. We can think of the origin as the artist's eye and the plane as her (infinite) canvas. Every (ordinary) point P on π corresponds to a unique line through the origin, namely the line OP. But this is a projective point according to our second point of view.



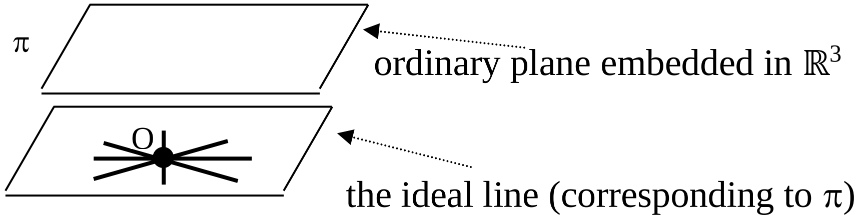
But if we take the totality of affine points the corresponding projective points (lines through O) do not use up all the available lines through the origin. Only a projective point that, viewed as a line through the origin, cuts π will correspond to a point on π (viz. the intersection of that line with π). Left over will be the projective points (lines through O) that are parallel to π. These are in fact the ideal points in the real projective plane.



Suppose we have two ordinary parallel lines on π. They don't intersect on π. But the corresponding projective lines *do* intersect. These corresponding projective lines will be the planes through the origin that pass through the respective lines on π. These lines will intersect in a line (just as two pages of a book intersect in the spine of the book).



This line will pass through O so it can be considered as a projective point. But it will be parallel to π and hence won't correspond to an ordinary point. So what were two parallel, non-intersecting lines in the ordinary plane π, correspond to two projective lines (planes through O) that *do* intersect – in an ideal point (line through O). The ideal points are in fact the lines through O that are parallel to π and the ideal line is thus the plane through O that is parallel to π.



In this way we have, in effect, completed the Real Affine Plane to produce the Real Projective Plane. But notice that the plane that becomes the ideal line depends on which affine plane we took in $\mathbb{R}^3$. In fact any projective line can become the ideal line if we take a suitable affine plane π and a suitable origin O.

This is important. The distinction between ordinary points and ideal points is not one that is intrinsic to the Real Projective Plane itself. It only reflects the way we view the Real Projective Plane from the point of view of a Real Affine Plane.

The beauty of the Real Projective Plane is that it is wonderfully uniform. We can prove Euclidean theorems by projective means without the messiness of having to consider cases where lines are parallel and where lines are not parallel. In the Real Projective Plane, all points are equivalent and all lines are equivalent.

Example 1: Suppose we take the plane π to be the one with equation $z = 1$.

(a) Which of the axes (x-, y- and z-) are ideal points and which are ordinary (relative to π)?
(b) What is the projective point corresponding to the ordinary point $(1, 1, 1)$ on π?
(c) What point on π corresponds to the projective point $x = 2y = 3z$?
(d) Relative to which planes is $x + y + z = 0$ the ideal line?

Solution: (a) The x- and y-axes are ideal, the z-axis is ordinary.
(b) The line $x = y = z$;
(c) (3, 3/2, 1)
(this is where the line $x = 2y = 3z$ intersects π);
(d) any plane of the form $x + y + z = c$ for $c \neq 0$.

§1.7. A Review of Relevant Linear Algebra

We're going to make this last view of the Real Projective Plane more rigorous by introducing linear algebra. Instead of talking about projective points as lines through the origin we shall think of them as one-dimensional subspaces of $\mathbb{R}^3$. This will free us from needing geometric intuition, and allow us to use only algebraic methods. But also we can change the underlying field of our vector space, giving rise to other projective planes – even finite ones. Before we do this we'll review the relevant linear algebra.

A **vector space** over a field F is a set (with the elements being called **vectors**) together with operations of addition and scalar multiplication (a **scalar** being an element of F) such that certain axioms are satisfied. Vectors are usually printed in bold type, such as **v**, to distinguish them from scalars (but there are times when this distinction can't be maintained).

Practically any mathematical object can be an element of a useful vector space. You can have vector spaces of numbers, of matrices, and of functions. But here we need to consider only those vectors of the form:

$$(x, y, z).$$

So our vectors will all have the form (x, y, z) with the components x, y, z coming from some field. We'll mainly use the field $\mathbb{R}$ of real numbers, though at one point we'll use various finite fields such as $\mathbb{Z}_p$, the field of integers modulo a prime p.

Vector addition and scalar multiplication are defined in the usual way, component by component.

A **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a vector of the form

$\lambda_1\mathbf{v}_1 + \ldots \lambda_n\mathbf{v}_n$ for some scalars $\lambda_1, \ldots, \lambda_n$.

A linear combination is **non-trivial** if at least one of the scalars is non-zero. A set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is **linearly independent** if $\lambda_1\mathbf{v}_1 + \ldots \lambda_n\mathbf{v}_n = \mathbf{0}$ implies that each $\lambda_i = 0$. Otherwise they are **linearly dependent**. In other words, a set of vectors is linearly dependent if there's a non-trivial linear combination of them that is zero.

In such a case one of the vectors can be written as a linear combination of the others. If they're linearly independent this isn't possible.

Note that it's the *set* that's linearly dependent or independent, not the vectors themselves. In other words it refers to the relationship between the vectors.

Example 2: Are the vectors:

$$(1, 5, -2), (6, -1, 9), (9, -17, 24)$$

linearly independent?

Solution: Sometimes students learn to answer such questions by putting the three vectors into a determinant and concluding that, because the determinant is zero, the vectors must be linearly dependent. That technique is valid in certain cases but it's best to go back to the definition and use the techniques of solving systems of linear equations. Apart from the fact that the determinant method only works for vectors in F^n, it's seen by many students to be a piece of hocus-pocus – a formula to be invoked, without any understanding. The following technique reveals what's happening, and need involve no more work.

Suppose that:

$$a(1, 5, -2) + b(6, -1, 9) + c(9, -17, 24) = (0, 0, 0).$$

This is equivalent to the system of equations:

$$\left.\begin{aligned} a + 6b + 9c &= 0 \\ 5a - b - 17c &= 0 \\ -2a + 9b + 24c &= 0 \end{aligned}\right\}$$

which can be represented by an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 6 & 9 & 0 \\ 5 & -1 & -17 & 0 \\ -2 & 9 & 24 & 0 \end{array}\right).$$

As is usual with systems of homogeneous equations we can omit the column of 0's and simply write the system as:

$$\begin{pmatrix} 1 & 6 & 9 \\ 5 & -1 & -17 \\ -2 & 9 & 24 \end{pmatrix}.$$

To solve this system we carry out a sequence of elementary row operations to put the matrix into echelon form. This simpler matrix represents a system of homogeneous linear equations that's equivalent to the original system (equivalent in the sense of having precisely the same solutions).

In this example we obtain:

$$\begin{pmatrix} 1 & 6 & 9 \\ 5 & -1 & -17 \\ -2 & 9 & 24 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 0 & -31 & -62 \\ 0 & 21 & 42 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 21 & 42 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 - 5R_1$, $R_3 + 2R_1$ $\quad$ $R_2 \div (-31)$ $\quad$ $R_3 - 21R_2$

Clearly this system has infinitely many solutions, and hence non-trivial ones. This is not so much because of the row of zeros at the bottom as the fact that we have an equivalent system with fewer equations than variables. In solving the above equivalent system:

$$\left.\begin{aligned} a + 6b + 9c &= 0 \\ b + 2c &= 0 \end{aligned}\right\}$$

we may choose *c* arbitrarily (and hence non-zero) and use the two equations to calculate the corresponding values of *a* and *b*.

The conclusion is that the vectors are linearly dependent. It's not necessary to exhibit an explicit relationship, but if called upon to do this we simply choose a convenient non-zero value of *c* and calculate the corresponding values of *a* and *b*.

For example if we take $c = 1$ we get $b = -2$ from the second equation and hence:

$$a = 12 - 9 = 3$$

from the first equation.

This gives the non-trivial linear relationship:

$$3(1, 5, -2) - 2(6, -1, 9) + (9, -17, 24) = (0, 0, 0),$$

which can be written as:

$$(9, -17, 24) = -3(1, 5, -2) + 2(6, -1, 9).$$

Example 3: Are the vectors:

$$(2, 5, 9), (3, 1, 0), (5, -2, 7)$$

linearly independent?

Solution: Writing them as the columns of a matrix (to get the coefficient matrix of the corresponding system of homogeneous linear equations) we have:

$$\begin{pmatrix} 2 & 3 & 5 \\ 5 & 1 & -2 \\ 9 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 5 & 1 & -2 \\ 9 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -\frac{13}{2} & -\frac{29}{2} \\ 0 & -\frac{27}{2} & -\frac{31}{2} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1 & \frac{3}{2} & \frac{5}{2}\\ 0 & 1 & \frac{29}{13}\\ 0 & -\frac{27}{2} & -\frac{31}{2}\end{pmatrix}\rightarrow\begin{pmatrix}1 & \frac{3}{2} & \frac{5}{2}\\ 0 & 1 & \frac{29}{13}\\ 0 & 0 & -\frac{31}{2}+\frac{27}{2}\cdot\frac{29}{13}\end{pmatrix}.$$

Since $-\frac{31}{2}+\frac{27}{2}\cdot\frac{29}{13}\neq 0$ this system has a unique solution (the zero one).

Note that if we're doing the calculations by hand we can avoid most fractions if we get leading 1's by subtracting suitable multiples of rows from others rather than by division. This greatly simplifies the arithmetic. Re-doing the above example we can obtain:

$$\begin{pmatrix}2 & 3 & 5\\ 5 & 1 & -2\\ 9 & 0 & 7\end{pmatrix}\rightarrow\begin{pmatrix}2 & 3 & 5\\ 1 & -5 & -12\\ 9 & 0 & 7\end{pmatrix}\rightarrow\begin{pmatrix}1 & -5 & -12\\ 2 & 3 & 5\\ 9 & 0 & 7\end{pmatrix}\rightarrow\begin{pmatrix}1 & -5 & -12\\ 0 & 13 & 29\\ 0 & 45 & 115\end{pmatrix}$$

$R_2 - 2R_1$ $\quad$ $R_1 \leftrightarrow R_2$ $\quad$ $R_2 - 2R_1$, $R_3 - 9R_1$

$$\rightarrow\begin{pmatrix}1 & -5 & -12\\ 0 & 13 & 29\\ 0 & 6 & 28\end{pmatrix}\rightarrow\begin{pmatrix}1 & -5 & -12\\ 0 & 1 & -27\\ 0 & 6 & 28\end{pmatrix}\rightarrow\begin{pmatrix}1 & -5 & -12\\ 0 & 1 & -27\\ 0 & 0 & 190\end{pmatrix}$$

$R_3 - 3R_2$ $\quad$ $R_2 - 2R_3$ $\quad$ $R_3 - 6R_2$

$$\rightarrow\begin{pmatrix}1 & -5 & -12\\ 0 & 1 & -27\\ 0 & 0 & 1\end{pmatrix} R_3 \div 190.$$

As before, this system has only the trivial solution so the vectors are linearly independent.

Example 4: Is the vector (3, 18, –4) a linear combination of the vectors (1, 5, –3) and (4, 22, 1)?

Solution: In other words, the question is asking whether the vectors are linearly dependent. We could proceed as in Example 3, but here's a slightly different method.

Are there scalars x, y such that:

$$(3, 18, -4) = x\,(1, 5, -3) + y\,(4, 22, 1).$$

We can write this equation as a non-homogeneous system of linear equations:

$$\left.\begin{aligned} x + 4y &= 3 \\ 5x + 22y &= 18 \\ -3x + y &= -4 \end{aligned}\right\}$$

Now we can write this system as an augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 4 & 3 \\ 5 & 22 & 18 \\ -3 & 1 & -4 \end{array}\right)$$ and, carrying out a sequence of elementary row operations, we put this in echelon form:

$$\left(\begin{array}{cc|c} 1 & 4 & 3 \\ 5 & 22 & 18 \\ -3 & 1 & -4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 13 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & -13 \end{array}\right)$$

$R_2 - 5R_1$, $R_3 + 3R_1$ $\quad$ $R_3 - 6R_2$

$$\to \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 1 & -13 \\ 0 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 1 & -13 \\ 0 & 0 & 29 \end{array}\right).$$

$R_2 \leftrightarrow R_3$ $\quad$ $R_3 - 3R_2$

This echelon form represents an equivalent system that's clearly inconsistent. In other words there's no solution for

x, y and so (3, 1, –4) is not a linear combination of the vectors (1, 5, 9) and (4, –2, 1).

A set of vectors **spans** a vector space V if every vector in V is a linear combination of them. A vector space is **finite-dimensional** if it has a finite spanning set. A set of vectors is a **basis** for V if it's linearly independent and also spans V.

If we have a finite spanning set for V we can remove suitable vectors one by one, as long as the set remains linearly dependent, without affecting the spanning, until we reach a basis. That is, every spanning set contains a basis. In a similar way, every linearly independent subset of V can be suitably extended to a basis.

The important fact about bases is that any two bases of a finite-dimensional vector space contain the *same number of vectors*. This unique number of vectors in a basis is called the **dimension** of the vector space.

For the proof of this fact I refer you to my notes on *Vector Spaces*. If you think it is pretty obvious let me point out that the analogous statements for other mathematical systems are not always true. For example, every finite abelian group is a direct sum of cyclic groups. These are analogous to one-dimensional subspaces. However the number of summands in such a direct sum is not unique. For example $\mathbb{Z}_{15} \oplus \mathbb{Z}_8 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_{120}$.

The vector space $\mathbb{R}^3$ clearly has dimension 3 since we have the basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)}.

If we have a vector space V of dimension *n*, then any set of more than *n* vectors must clearly be linearly dependent (otherwise we could extend them to a basis with more vectors than the dimension). And if we have a set of less than *n* vectors they can't possibly span V (otherwise the dimension would be less than *n*).

Example 5: Are the vectors (1, 4, 6), (2, 5, 1), (6, 1, 9), (4, 8, 3) linearly independent?

Solution: The answer is clearly NO since you can't have a set of four linearly independent vectors inside a 3-dimensional vector space. There's no need to do any calculation in this case other than to count the number of vectors!

A vector space V is said to be the **sum** of subspaces U and W if every **v** ∈ V can be expressed as **v** = **u** + **w** where **u** ∈ U and **w** ∈ W. It is said to be the **direct sum** if, as well, U ∩ W = 0. In the first case we write V = U + W and in the case of direct sum we write V = U ⊕ W. The special feature of a direct sum is that if V = U ⊕ W then every vector **v** ∈ V can be expressed *uniquely* as **v** = **u** + **w** where **u** ∈ U and **w** ∈ W.

If we take a basis for each of the subspaces in a direct sum, and combine them, we'll produce a basis for the whole space. This means that:

$$\dim(U \oplus V) = \dim U + \dim V.$$

The **dot product** of two vectors in F^n is defined by:

$$(x_1, \ldots, x_n).(y_1, \ldots, y_n) = x_1y_1 + \ldots + x_ny_n.$$

The **orthogonal complement** of a subspace U of a vector space V is defined to be the set of all vectors that are orthogonal to every vector in U, that is:

$$\mathbf{U}^\perp = \{\mathbf{v} \in V \mid \mathbf{u}.\mathbf{v} = 0 \text{ for all } \mathbf{u} \in U\}.$$

It's easy to show that this is a subspace of V.

If the vector space is $\mathbb{R}^3$ we can view the dot product geometrically and orthogonality is virtually synonymous with perpendicularity. (The one exception is that the zero vector is orthogonal to every vector, but it could hardly be called perpendicular.)

In $\mathbb{R}^3$ the orthogonal complement of a 1-dimensional subspace, that is a line through the origin, is the plane through the origin that is perpendicular to it. And the orthogonal complement of a 2-dimensional subspace, that is a plane through the origin, is the line that is perpendicular to it. Hence $\mathbb{R}^3 = U \oplus U^\perp$ for all subspaces U. Over other fields this may not be the case.

Example 6: Over $\mathbb{Z}_2$, if $U = \langle(1, 1, 0)\rangle$ then $U \leq U^\perp$ and so

$$U + U^\perp = U^\perp.$$

So we must be careful not to assume that U and $U^\perp$ are always vector space complements (even though we call $U^\perp$ the orthogonal complement). However the following theorems will hold for any field.

Theorem 2: If U is a subspace of F^n then:

$$\dim U + \dim U^{\perp} = n.$$

Proof: Suppose that U has a basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ and let $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ for each i.

Let $A = (a_{ij})$ be the $m \times n$ matrix where each $a_{ij} \in F$.

Then $\dim U = \text{rank}(A)$ and $\dim U^{\perp} = \text{nullity}(A)$.

Since $\text{rank}(A) + \text{nullity}(A) = n$, the result follows. ☺✋

Theorem 3: If $U \leq F^n$ then $U = U^{\perp\perp}$.

Proof: Clearly $U \leq U^{\perp\perp}$.

Now $\dim U + \dim U^{\perp} = n = \dim U^{\perp} + \dim U^{\perp\perp}$ and so:

$$\dim U = \dim U^{\perp\perp}.$$

It follows that $U = U^{\perp\perp}$. ☺✋

Theorem 4: In F^n, $U \leftrightarrow U^{\perp}$ is a 1-1 correspondence between subspaces of dimension r and subspaces of dimension $n - r$.

Proof: Clearly if $U = V$ then $U^{\perp} = V^{\perp}$.

Now suppose that $U^{\perp} = V^{\perp}$.

Hence $(U^{\perp})^{\perp} = (V^{\perp})^{\perp}$, that is, $U = V$. ☺✋

Another concept from linear algebra that we'll find useful is that of a linear transformation.

A function $F: U \to V$ is a **linear transformation** if:

$$F(\lambda\mathbf{u} + \mu\mathbf{v}) = \lambda F(\mathbf{u}) + \mu F(\mathbf{v})$$

for all **u**, **v** in U and all scalars λ and μ.

The **kernel** of such a linear transformation is:

$$\mathbf{ker\ F} = \{\mathbf{u} \in U \mid F(\mathbf{u}) = \mathbf{0}\}$$

and its **image** is:

$$\mathbf{im\ F} = \{F(\mathbf{u}) \mid \mathbf{u} \in U\}.$$

The kernel is a subspace of U and its dimension is called the **nullity** of F. The image is a subspace of V and its dimension is called the **rank** of F. One can show that

$$\text{rank}(f) + \text{nullity}(f) = \dim U.$$

For projective planes we will be dealing with F^3. We define the projective points and planes as 1- and 2-dimensional subspaces respectively.

§1.8. The Algebraic Version of the Projective Plane

The description of the Real Projective Plane as lines and planes through the origin is better than the intuitive one, where we simply invented ideal points. This is because it frees us from having to maintain the artificial distinction between the ordinary and the ideal. But it's still somewhat difficult to work with because it involves 3-dimensional geometry. The effective way of dealing with this is to use linear algebra. This third development of the Real Projective Plane is essentially the same as the previous one, except that things are expressed algebraically instead of geometrically.

Although we're focussing on the Real Projective Plane we can do projective geometry for any field. We work in the 3-dimensional vector space F^3. The vectors have the form (x, y, z) where x, y and z belong to the field.

A **projective point** is a 1-dimensional subspace of F^3.
A **projective line** is a 2-dimensional subspace of F^3.
A projective point P **lies on** a projective line *h* if the 1-dimensional subspace, P, is a subspace of the 2-dimensional subspace *h*. (Alternatively we say that *h* **passes through** P.)

Theorem 1C: The Real Projective Plane is complete. That is:
(1) any two projective points lie on exactly one projective line.
(2) any two projective lines intersect in exactly one projective point.
Proof:
(1) Let $P = \langle \mathbf{p} \rangle$ and $Q = \langle \mathbf{q} \rangle$ be two distinct projective points. Then $P + Q = \langle \mathbf{p}, \mathbf{q} \rangle$ is clearly the only 2-dimensional subspace containing both P and Q. It is thus the only projective line passing through the projective points P, Q.
(2) Let *h*, *k* be two distinct projective lines.
Now $h + k$ contains both *h* and *k*.
If $h + k = h$ then $k \subseteq h + k = h$. Since *h*, *k* both have dimension 2 we must have $h = k$. [Recall that if two subspaces have the same dimension and one is inside the other, they must be equal.] This contradicts our assumption.

Hence *k* is a proper subspace of $h + k$, so:

$$\dim(h + k) = 3.$$

Now from linear algebra we have:

$$\dim(h + k) = \dim(h) + \dim(\mathrm{k}) - \dim(h \cap k).$$

Therefore $3 = 2 + 2 - \dim(h \cap k)$, and so:

$$\dim(h \cap k) = 1.$$

So $h \cap k$ is a projective point, and being their intersection, it lies on both h and k. ☺✋

Theorem 5: Three projective points $\mathrm{P} = \langle \mathbf{p} \rangle$, $\mathrm{Q} = \langle \mathbf{q} \rangle$ and $\mathrm{R} = \langle \mathbf{r} \rangle$ are collinear if and only if the vectors **p**, **q** and **r** are linearly dependent.

Proof: For $\langle \mathbf{p} \rangle$, $\langle \mathbf{q} \rangle$, $\langle \mathbf{r} \rangle$ to be collinear they must lie on a common projective line. That means they must lie in a 2-dimensional subspace. For three vectors to lie in a 2-dimensional subspace they must be linearly dependent. Conversely if **p**, **q** and **r** are linearly dependent they'll span a subspace of dimension at most 2. If $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ has dimension 2 then it's a projective line passing through the projective points P, Q, R. ☺✋

Remember that in the case of $\mathbb{R}^3$ we have another product of vectors – the **cross product**. This can be defined for any field, even if we can't visualize it. For any two vectors $\mathbf{u}, \mathbf{v} \in \mathrm{F}^3$ we define $\mathbf{u} \times \mathbf{v}$ by:

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$= (x_2y_3 - x_3y_2, \; -(x_1y_3 - x_3y_1), \; x_1y_2 - x_2y_1).$$

It's more easily remembered as the 3×3 'determinant':

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The top row are the standard basis vectors:

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

and not scalars like the second and third rows. Nevertheless, using the usual rules for expanding determinants, we obtain the correct expression for the cross-product.

Example 7:

If $\mathbf{u} = (1, 5, -3)$ and $\mathbf{v} = (4, -1, 2)$ find $\mathbf{u} \times \mathbf{v}$.

Solution: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & -3 \\ 4 & -1 & 2 \end{vmatrix}$

$= [5.2 - (-3)(-1)]\,\mathbf{i} - [1.2 - (-3).4)]\,\mathbf{j} + [1.(-1) - 5.4]\,\mathbf{k}$

$= 7\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} = (7, -14, -21)$.

Check: $(7, -14, -21).(1, 5, -3) = 7 - 70 + 63 = 0$.

$(7, -14, -21).(4, -1, 2) = 28 + 14 - 42 = 0$.

Since it's possible to make an error in sign at some stage it's important that you check your answer to a cross-product against at least one of the two vectors.

Theorem 6: $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

Proof: Let $\mathbf{u} = (x_1, x_2, x_3)$ and $\mathbf{v} = (y_1, y_2, y_3)$.

Then $\mathbf{u} \times \mathbf{v} = (x_2y_3 - x_3y_2,\ \ -(x_1y_3 - x_3y_1),\ \ x_1y_2 - x_2y_1)$.

Hence $(\mathbf{u} \times \mathbf{v}).\mathbf{u}$

$= (x_2y_3 - x_3y_2)x_1 - (x_1y_3 - x_3y_1)x_2 + (x_1y_2 - x_2y_1)x_3$

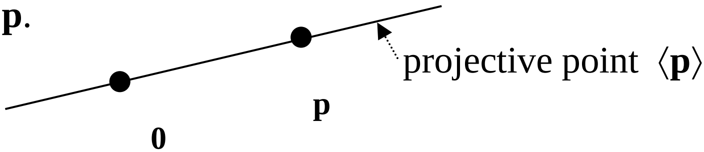
$= x_2y_3x_1 - x_3y_2x_1 - x_1y_3x_2 + x_3y_1x_2 + x_1y_2x_3 - x_2y_1x_3$

$= x_1x_2y_3 - x_1x_3y_2 - x_1x_2y_3 + x_2x_3y_1 + x_1x_3y_2 - x_2x_3y_1$

$= 0$.

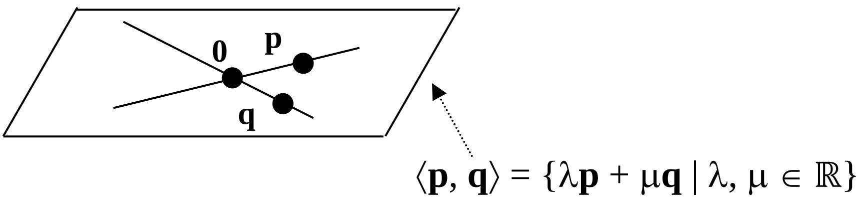
Similarly, $(\mathbf{u} \times \mathbf{v}).\mathbf{v} = 0$. ☺🖐

Now a projective point, P, being a 1-dimensional subspace, has a basis consisting of one non-zero vector **p** so that P = ⟨**p**⟩. The elements of P are scalar multiples of **p**.



Note that if P = ⟨**p**⟩ and Q = ⟨**q**⟩ are projective points then P = Q doesn’t necessarily imply that **p** = **q**. What it does mean is that each is a non-zero scalar multiple of the other, or to use the language of vector spaces, the set {**p**, **q**} is linearly dependent. On the other hand if P, Q are distinct points then **p**, **q** are linearly independent. (This discussion applies over any field but the picture only makes sense for the Real Projective Plane.

If P = ⟨**p**⟩ and Q = ⟨**q**⟩ are distinct then the smallest subspace containing them both is the 2-dimensional subspace ⟨**p**, **q**⟩. This is the line projective line PQ.



If $\mathbf{u} \times \mathbf{v} = 0$ then **u**, **v** are linearly dependent. If the field is ℝ then we can prove this geometrically using the result that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|.|\mathbf{v}|.\sin\theta$ where θ is the angle between **u**, **v**. This doesn't make sense over a finite field since angles can't be defined and non-zero vectors can have zero length.

Example 8: The length of (1, 1, 1) in $\mathbb{Z}_3^3$ is zero.

Theorem 6: If $\mathbf{u}, \mathbf{v} \in F^3$ are linearly independent, then:

$$\mathbf{u} \times \mathbf{v} \neq \mathbf{0}.$$

Proof: Suppose that $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (d, e, f)$ are linearly independent.

Now $\mathbf{u} \times \mathbf{v} = (bf - ce, cd - af, ae - bd)$.

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then:

$$\left.\begin{array}{r} bf = ce \\ cd = af \\ ae = bd \end{array}\right\}$$

If $a \neq 0$, $e = \dfrac{bd}{a}$ and $f = \dfrac{cd}{a}$.

Hence $a\left(1, \frac{b}{a}, \frac{c}{a}\right) = (a, b, c)$ and

$d\left(1, \frac{b}{a}, \frac{c}{a}\right) = \left(d, \frac{bd}{a}, \frac{\text{cd}}{\text{a}}\right) = (d, e, f)$.

Thus **u**, **v** are linearly dependent, a contradiction.

Suppose now that $a = 0$.

Then $\left.\begin{array}{r} bf = ce \\ cd = 0 \\ bd = 0 \end{array}\right\}$. If $d \neq 0$ then $b = c = 0$, a contradiction.

Hence $d = 0$. So $\mathbf{u} = (0, b, c)$ and $\mathbf{v} = (0, e, f)$.

Since $bf = ce$, **u**, **v** are linearly dependent. ☺🖐

You may feel that it would have been simpler to use the fact that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|.|\mathbf{v}|.\sin\theta$. If u, v are linearly independent, doesn't this show that $|\mathbf{u} \times \mathbf{v}| \neq 0$? Certainly this is the case if the field is $\mathbb{R}$, but there would be difficulties over other fields. We need to stick to algebraic methods.

Theorem 7: In F^3, if $\langle \mathbf{p} \rangle \neq \langle \mathbf{q} \rangle$, $\langle \mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{p} \times \mathbf{q} \rangle^{\perp}$.
Proof: Since $\langle \mathbf{p} \rangle \neq \langle \mathbf{q} \rangle$, $\dim \langle \mathbf{p}, \mathbf{q} \rangle = 2$, so:

$$\dim \langle \mathbf{p}, \mathbf{q} \rangle^{\perp} = 1.$$

Since $\mathbf{p} \times \mathbf{q} \in \langle \mathbf{p}, \mathbf{q} \rangle$, and $\mathbf{p} \times \mathbf{q} \neq 0$ then:

$$\langle \mathbf{p}, \mathbf{q} \rangle^{\perp} = \langle \mathbf{p} \times \mathbf{q} \rangle.$$

Hence $\langle \mathbf{p}, \mathbf{q} \rangle = \langle \mathbf{p}, \mathbf{q} \rangle^{\perp\perp} = \langle \mathbf{p} \times \mathbf{q} \rangle^{\perp}$. ☺🖐

EXERCISES FOR CHAPTER 1

Exercise 1: For each of the following statements determine whether it is true or false.

(1) An ordinary point cannot lie on the ideal line.

(2) An ideal point cannot lie on an ordinary line.

(3) Every ordinary line cuts the ideal line.

(4) Two non-parallel ordinary lines cannot intersect in an ideal point.

(5) There are infinitely many lines parallel to the ideal line.

(6) There are infinitely many parallel classes.

(7) There are infinitely many ideal points.

(8) There are infinitely many ideal lines.

(9) There is exactly one ideal point on every projective line.

(10) There is exactly one ideal line through every projective point.

Exercise 2: Which of the following sets of vectors are linearly dependent?

With $\mathbb{R}$ as the field of scalars:

(a) {(3, 4, 7), (2, 8, 3), (10, 8, 25)};

(b) {(4, 6, 2), (6, 0, −1), (5, 1, 3), (1, 1, 0)};

(c) {(6, 9, 11), (2, 1, 1)};

(d) {(12, 15, 9), (20, 25, 15)}.

With $\mathbb{Z}_2$ as the field of scalars:

(e) {(1, 1, 1), (1, 1, 0), (0, 0, 1)};

(f) {(1, 0, 0), (1, 1, 0), (0, 1, 1), (1, 1, 1)}.

Exercise 3: Write (10, 8, 25) as a linear combination of the vectors (3, 4, 7) and (2, 8, 3).

Exercise 4:
(a) What is the dimension of the subspace of $\mathbb{R}^3$ spanned by (3, 4, 7), (2, 8, 3) and (10, 8, 25)?
(b) What is the dimension of the space of all vectors which are orthogonal to all three of the vectors (3, 2, 7), (1, 1, 0), (2, 3, –7)?
(c) With $\mathbb{Z}_5$ as the field of scalars find the dimension of the subspace spanned by the vectors:

(1, 3, 2), (4, 1, 0) and (0, 1, 3).

Exercise 5: If **a** = (6, 3, 2) and **b** = (–2, 1, 5), find:

(a) **a.a**;
(b) **a.b**;
(c) **b.a**;
(d) **b.b**;
(e) $\mathbf{a} \times \mathbf{a}$;
(f) $\mathbf{a} \times \mathbf{b}$;
(g) $\mathbf{b} \times \mathbf{a}$;
(h) $\mathbf{b} \times \mathbf{b}$.

Exercise 6: (a) Prove that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}).$$

(b) Hence show that for all $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

Exercise 7: Find $\langle(4, -7, 8)\rangle^{\perp} \cap \langle(3, 1, -2)\rangle^{\perp}$.

Exercise 8: Let $\mathbf{a} = (1, 2, 3)$. Find the rank and nullity of the following (where they exist):
(a) $F: \mathbb{R}^3 \to \mathbb{R}$ defined by $F(\mathbf{v}) = \mathbf{v}.\mathbf{a}$.
(b) $G: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $G(\mathbf{v}) = \mathbf{v} \times \mathbf{a}$.
(c) $H: \mathbb{R}^3 \to \mathbb{R}$ defined by $H(\mathbf{v}) = \mathbf{v}.\mathbf{v}$.

Exercise 9: If $A = \langle(1, 2, 3)\rangle$ and $B = \langle(5, 1, 5)\rangle$ find AB.

Exercise 10: If $P = \langle(1, 1, 1)\rangle$ and $Q = \langle(1, -2, 3)\rangle$ does $R = \langle(3, 1, 4)\rangle$ lie on PQ?

Exercise 11: If $A = \langle\mathbf{a}\rangle$, $B = \langle\mathbf{b}\rangle$ and $C = \langle\mathbf{a} + \mathbf{b}\rangle$, show that C lies on AB.

Exercise 12: Let $P = \langle(1, 1, 1)\rangle$, $Q = \langle(1, 2, 3)\rangle$,
$R = \langle(2, 3, 1)\rangle$ and $S = \langle(3, 1, 2)\rangle$. Find $PQ \cap RS$

Exercise 13: Let $A = \langle(4, -10, 5)\rangle$, $B = \langle(0, 7, -3)\rangle$,
$C = \langle(1, 1, 0)\rangle$ and $D = \langle(1, 2, 2)\rangle$.
Let $M = AB \cap CD$ and let $N = AC \cap BD$. Find MN and determine whether $AD \cap BC$ lies on MN.

Exercise 14: Is $\langle(1, 0, 1)\rangle$ an ideal point?

Exercise 15: If we embed the Euclidean plane in the projective plane by taking the plane
$\pi = \{(x, y, z) \mid x + 2y + 3z = 4\}$, determine whether
$\langle(1, -2, 1)\rangle$ is an ideal point.

SOLUTIONS FOR CHAPTER 1

Exercise 1:
(1) TRUE
(2) FALSE
(3) TRUE
(4) FALSE
(5) FALSE
(6) TRUE
(7) TRUE
(8) FALSE
(9) FALSE: Don't forget the ideal line
(10) FALSE.

Exercise 2: (a) Suppose:

$$x(3, 4, 7) + y(2, 8, 3) + z(10, 8, 25) = (0, 0, 0).$$

Then x, y, z satisfy the system of homogeneous linear equations with coefficient matrix:

$$\begin{pmatrix} 3 & 2 & 10 \\ 4 & 8 & 8 \\ 7 & 3 & 25 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 3 & 2 & 10 \\ 1 & 6 & -2 \\ 7 & 3 & 25 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 6 & -2 \\ 3 & 2 & 10 \\ 7 & 3 & 25 \end{pmatrix} \xrightarrow{R_2 - 3R_1,\ R_3 - 7R_1} \begin{pmatrix} 1 & 6 & -2 \\ 0 & -16 & 16 \\ 0 & -39 & 39 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$R_2 \div (-16)$, $R_3 + 39R_2$

This system has the non-trivial solution:

$$z = 1,\ y = 1,\ x = -4.$$

Hence 4 (3,4,7) = (2, 8, 3) + (10, 8, 25) and so the set is linearly **dependent**.

(b) The set is linearly **dependent** since there are 4 vectors in a 3-dimensional vector space.

(c) A set of two vectors is linearly dependent if and only if one is a scalar multiple of the other. This is not the case here so the set is linearly **independent**.

(d) (12, 15, 9) = (3/5)(20, 25, 15) so the set is linearly **dependent**.

(e) The sum of the vectors is **0** so the set is linearly **dependent**.

(f) The set is linearly **dependent** for the same reason as in (b)

Exercise 3: Let $x(3, 4, 7) + y(2, 8, 3) = (10, 8, 25)$.

Then x, y satisfy the system of linear equations:

$$\left(\begin{array}{cc|c} 3 & 2 & 10 \\ 4 & 8 & 8 \\ 7 & 3 & 25 \end{array}\right).$$

Proceeding as in Exercise 3(a) we get the echelon form

$$\left(\begin{array}{cc|c} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$ which gives $y = -1$, $x = 4$.

Hence (10, 8, 25) = **4 (3, 4, 7) – (2, 8, 3)**.

Exercise 4:

(a) The vectors are linearly dependent, as shown in Exercise 2(a), so the dimension less than 3. As they are not all scalar multiples of the same vector the dimension is greater than 1. Hence the dimension = **2**.

(b) Suppose that:

$$x(3, 2, 7) + y(1, 1, 0) + z(2, 3, -7) = (0, 0, 0).$$

Then x, y, z satisfy the homogeneous system:

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 7 & 0 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 7 & 0 & -7 \end{pmatrix}$$. Since $R_3 = 7R_1$ this system has a non-zero solution and so the set of vectors is linearly dependent. So they span a 2-dimensional subspace and their orthogonal complement has dimension 3 – 2 = **1**.

(c) Suppose $x(1, 3, 2) + y(4, 1, 0) + z(0, 1, 3) = (0, 0, 0)$.

Then x, y, z satisfy the homogeneous system:

$$\begin{pmatrix} 1 & 4 & 0 \\ 3 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

$R_2 - 3R_1$, $R_3 - 2R_1$ $\quad R_2 \div 4$ $\quad R_3 - 2R2$

This system has only the zero solution so the set of vectors is linearly independent. So they span a **3-** dimensional subspace.

Exercise 5:

(a) $\mathbf{a.a} = 36 + 9 + 4 = 49$;

(b) $\mathbf{a.b} = -12 + 3 + 10 = 1$;

(c) $\mathbf{b.a} = 1$ (always $\mathbf{b.a} = \mathbf{a.b}$);

(d) $\mathbf{b.b} = 4 + 1 + 25 = 30$;

(e) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ (true for all $\mathbf{a}$);

(f) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & 2 \\ -2 & 1 & 5 \end{vmatrix} = \mathbf{(13, -34, 12)}$

(check: $78 - 102 + 24 = 0$); $\mathbf{b} \times \mathbf{a} = \mathbf{(-13, 34, -12)}$

(always $\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}$);

(g) $\mathbf{b} \times \mathbf{b} = \mathbf{0}$ (always).

Exercise 6:

(a) The computations of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ as 3×3 determinants are identical, except for the swapping of the 2^{nd} and 3^{rd} rows. This changes the sign of a determinant.

(b) By (a) $\mathbf{v} \times \mathbf{v} = -\,\mathbf{v} \times \mathbf{v}$, so $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

Exercise 7:

$(x, y, z) \in \langle(4, -7, 8)\rangle^{\perp} \cap \langle(3, 1, -2)\rangle^{\perp}$ if and only if it is orthogonal to both vectors. Such vectors are scalar multiples of the cross product.

$$(4, -7, 8) \times (3, 1, -2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -7 & 8 \\ 3 & 1 & -2 \end{vmatrix} = (6, 32, 25).$$

CHECK: 4.6 – 7.32 + 8.25 = 24 – 224 + 200 = 0.
So the answer is **⟨(6, 32, 25)⟩**.

Exercise 8:
(a) F is a linear transformation and ker F is the orthogonal complement of ⟨(1, 2, 3)⟩.
Hence dim ker F = 2. Thus the **nullity = 2** and the **rank is thus 1** (ie 3 – 2).

(b) G is a linear transformation.
ker G = {**v** | **v** × **a** = **0**} = ⟨**a**⟩.
So the **nullity is 1** and the **rank is 2**.

(c) H is **not a linear transformation** and so rank and nullity are not defined.
[eg H(2**v**) = 2**v**.2**v** = 4(**v**.**v**) = 4H(**v**), not 2H(**v**)]

Exercise 9: AB = $\langle(1, 2, 3) \times (5, 1, 5)\rangle^{\perp} = \langle(7, 10, -9)\rangle^{\perp}$.

Exercise 10: PQ = $\langle(5, -2, -3)\rangle^{\perp}$.
Since (3, 1, 4).(5, –2, –3) = 15 –2 – 12 = 1 ≠ 0, R does not lie on PQ. (Alternatively we can put the vectors for P, Q and R as the rows of a 3 × 3 matrix and show that the determinant is non-zero).

Exercise 11: Since **a**, **b**, **a** + **b** are linearly dependent, C lies on AB.

Exercise 12:

PQ = $\langle(1, 1, 1) \times (1, 2, 3)\rangle = \langle(1, -2, 1)\rangle^{\perp}$,
RS = $\langle(2, 3, 1) \times (3, 1, 2)\rangle = \langle(5, -1, -7)\rangle^{\perp}$,
PQ $\cap$ RS = $\langle(1, -2, 1) \times (5, -1, -7)\rangle$
$= \langle(15, 12, 9)\rangle = \langle(5, 4, 3)\rangle$.

Exercise 13:

AB = $\langle(-5, 12, 28)\rangle^{\perp}$ and CD = $\langle(2, -2, 1)\rangle^{\perp}$.
If M = $\langle\mathbf{m}\rangle$ then **m** is orthogonal to both (–5, 12, 28) and (2, –2, 1) and so is a scalar multiple of:

$$(-5, 12, 28) \times (2, -2, 1) = (68, 61, -14).$$

Hence M = $\langle(68, 61, -14)\rangle$.
Similarly AC = $\langle(-5, 5, 14)\rangle^{\perp}$ and BD = $\langle(20, -3, -7)\rangle^{\perp}$ and N = $\langle(7, 245, -85)\rangle$.
Then MN = $\langle(-1755, 5682, 16233)\rangle$.
Now AD = $\langle(-30, -3, 18)\rangle^{\perp}$ and BC = $\langle(3, -3, -7)\rangle^{\perp}$ so AD $\cap$ BC = $\langle(-4155, -2424, -7329)\rangle$.
It would lie on MN if the vectors (–1755, 5682, 16233) and (–4155, –2424, –7329) were orthogonal. Since their dot product is $-125452800 \neq 0$ they are not orthogonal and so AD $\cap$ BC does not lie on MN.

Exercise 14: Sorry to trick you! Of course there’s no distinction between ideal points and ordinary points in the projective plane itself. The concepts only make sense when we embed a Euclidean plane in the projective plane. Depending on which embedding we take this point could be ordinary or ideal.

Exercise 15: Since (1, –2, 1) lies on the plane $x + 2y + 3z = 0$ the line joining the origin to (1, –2, 1) is parallel to π and so does not cut π. Hence ⟨(1, –2, 1)⟩ is an ideal point.